Spectral Theorem: Let S be a (real) symmetric matrix

Then $S = QDQ^T$ where D is diagonal

and Q is an orthogonal matrix of eigenvectors.

(S is orthogonally diagonalizable)

To prove this, we need two key ingredients.

Key Property 1: A symmetric matrix has only real eigenvalues.

Proof:

Notation: If $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$, then $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$.

Hence $\bar{x}^T x = \bar{x}_1 \cdot x_1 + \cdots + \bar{x}_n x = \|x_1\|^2 + \cdots + \|x_n\|^2 > 0$ for $x \neq 0$

Let $\lambda \in \mathbb{C}$ be an eigenvalue of S. We must show $\lambda \in \mathbb{R}$.

Let $Sx = \lambda x$ for some nonzero $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}$

We want to show $\lambda \in \mathbb{R}$. Conjugating $Sx = \lambda x$ gives

$$\overline{Sx} = \overline{\lambda x} \Rightarrow \bar{S}\bar{x} = \bar{\lambda}\bar{x} \Rightarrow S\bar{x} = \bar{\lambda}\bar{x} \text{ since } S \text{ is real}$$

$$\Rightarrow (S\bar{x})^T = (\bar{\lambda}\bar{x})^T$$

$$\Rightarrow \bar{x}^T S^T = \bar{\lambda}\bar{x}^T$$

$$\Rightarrow \bar{x}^T S = \bar{\lambda}\bar{x}^T$$

$$\Rightarrow \bar{x}^T S x = \bar{\lambda}\bar{x}^T x$$

On the other hand, $Sx = \lambda x \Rightarrow \bar{x}^T S x = \bar{x}^T(\lambda x) = \lambda \bar{x}^T x$

So $\bar{\lambda}\bar{x}^T x = \lambda \bar{x}^T x$. Since $\bar{x}^T x > 0$, we have

$\bar{\lambda} = \lambda \Rightarrow \lambda \in \mathbb{R}$. ∎

Key Property 2: The eigenspaces of a symmetric matrix S are orthogonal.

Proof: Let λ_1, λ_2 be eigenvalues of S.

Let $x_1 \in E_{\lambda_1}(S)$ and $x_2 \in E_{\lambda_2}(S)$.

We need to show that $x_1 \perp x_2$

Since $E_{\lambda_1}(S) = N(S-\lambda_1 I)$, we have

$x_1 \in N(S-\lambda_1 I) = N(S^T - \lambda_1 I^T) = N(S-\lambda_1 I)^T)$ left nullspace

Since $N((S-\lambda_1 I)^T) \perp C(S-\lambda_1 I)$, x_1 is orthogonal to all elements in the column space $C(S-\lambda_1 I)$

So if we show $x_2 \in C(S-\lambda_1 I)$, we are done.

Notice $(S-\lambda_1 I)x_2 = Sx_2 - \lambda_1 I x_2 = \lambda_2 x_2 - \lambda_1 x_2 = (\lambda_2 - \lambda_1)x_2$

Since $\lambda_1 \neq \lambda_2$, we have $x_2 = (S-\lambda_1 I)\left(\frac{1}{\lambda_1 - \lambda_2} x_2\right)$

$\Rightarrow x_2 \in C(S-\lambda_1 I) \Rightarrow x_1 \perp x_2$. ■

Proof of Spectral Thm:

Case 1: n distinct real eigenvalues

n distinct evalues $\Rightarrow$ n orthonormal evectors (Key Prop. 2)

$\Rightarrow$ n L.I. evectors (from 10/22 Worksheet)

$\Rightarrow$ by diagonalizability thm, $S = PDP^{-1}$, where where P is orthogonal $\Rightarrow S = PDP^T$.

Case 2: eigenvalues not distinct.

This is more complicated. See text for the idea. ■

Positive Definite Matrices

We've seen that symmetric matrices have special properties. Symmetric matrices that are "positive definite" are even more special and arise in many applications.

Def: A symmetric matrix S is:

positive definite if its eigenvalues are positive.

positive semi definite if its eigenvalues are nonnegative

Ex: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is positive semidefinite but not positive definite since its eigenvalues are $(0,1,2)$

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has eigenvalue $-1 \Rightarrow$ not positive semi definite.

Proposition: S is positive definite (resp. positive semi-definite) if and only if $x^TSx > 0 \quad \forall\, x \neq 0$ (rep. $x^TSx \geq 0 \quad \forall\, x$)

Key Example: If A is any matrix, then A^TA is positive semidefinite

Proof: Since $x^T(A^TA)x = (x^TA^T)(Ax) = (Ax)\cdot(Ax) \geq 0 \quad \forall\, x \in \mathbb{R}^n$

A^TA is positive semi-definite by the proposition ∎

Useful observation:

Recall, A^TA is invertible $\Leftrightarrow$ A has L.I. columns.

Moreover from HW, A^TA is invertible $\Leftrightarrow$ its eigenvalues are nonzero

Hence A^TA is positive definite $\Leftrightarrow$ A has L.I. columns